Rederivation of diffraction calculations of a tenuous isothermal atmosphere

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2006 March 28

1 Purpose

We are finishing the paper on the C313.2 observations, and want to strictly limit the amount of atmosphere around Charon. The reason is twofold. First, there is inherent interest in the presense or absense of an atmosphere around Charon. Second, the presense of an atmosphere may affect the derived radius.

The solution appears to be contained in French and Gierasch 1976, "Diffraction calculation of occultation lightcurves in the presence of an isothermal atmosphere," Astron. J. 81, 445-451, equation 7. However, this appears to have typographical errors.

We are looking for a solution that (1) reproduces Fresnel diffraction in the airless case, (2) reproduces Baum and Code refractive lightcurves in the case of no edge.

2 Preliminaries

We summarize the relevant starting points.

First, the definitions in FG76

$\lambda =$ wavelength of observation

$D =$ distance to occulting body

$l = (\lambda D/2)^{1/2}$ Fresnel scale

H scale height

R radius of occulting body

ϕ_0 phase excess of graxing ray in radians

ν_0 refractivity of the atmosphere at the surface

$b = \phi_0 l^2/\pi H^2$ differential bending parameter

y	distance along lightcurve from shadow boundary in units of l
y_0	location along light curve of line pasing through occulted object and planetary limb.

We can relate b and y_0 to more usual non-diffractive occultation terminology by noting that since $\phi_0 = \nu_0(2\pi)^{3/2}(RH)^{1/2}/\lambda$, and since the bending angle (let's call it αsince RG76 use θ for the total exponent in the Fresnel convolution) is related to $\nu_0$by

$$\alpha(r) = \frac{d}{dr}\int_{-\infty}^{\infty} \nu((r^2 + x^2)^{1/2})dx$$

Since $\nu$is exponential with scale height , $\alpha = \sqrt{2\pi R/H}\nu$. In particular, the bending angle for the surface-grazing ray, α_0, is $\alpha_0 = \sqrt{2\pi R/H}\nu_0$. We also have $y_0 = (D/l)\alpha_0 = \phi_0 l/\pi H$.

We can simplify our lives by defining a few more variables:

$h = H/l$	(scale height in units of l)
$d = D/l$	(distance to occulting body in units of l)
$r = R/l$	(radius of occulting body in units of l)

Also, I'll be using $\phi_s$instead of $\phi_0$for the surface phase excess, and w_s instead of y_0 for the location of the surface in the plane of the phase screen.

3 Preliminaries

We start with the standard equations for Fresnel diffraction by a thin screen. Since we're following FG76, we use their version (more or less).

$$I(y) = KK^*/2$$

$$K(y) = \int e^{i\theta(w)}dw$$

$$\theta(w) = \frac{\pi}{2}(w-y)^2 + \Phi(w)$$

$$\Phi(w) = (2\pi)^{3/2}\frac{(RH)^{1/2}}{\lambda}\nu_0$$

The basic trick, as with most Fresnel calculations, is to turn the exponent into something quadratic in w. This can be then turned into expressions involving the Fresnel integrals

$$C(z) = \int_0^z \cos(\pi z^2/2) dt$$

$$S(z) = \int_0^z \sin(\pi z^2/2) dt$$

Note than $-C(-z) = C(z)$, $-S(-z) = S(z)$, and $C(\infty) = S(\infty) = 1/2$. This all means that

$$\int_a^b e^{i(\pi/2)z^2} dz = [C(b) + C(-a)] + i\,[S(b) + S(-a)]$$

and

$$\int_a^\infty e^{i(\pi/2)z^2} dz = \left[\frac{1}{2} + C(-a)\right] + i\left[\frac{1}{2} + S(-a)\right]$$

It's useful to notice that constant terms (terms that don't depend on w) in θ don't affect the intensity.

$$K(y) = \int e^{i\theta(w)+\psi(y)} dw = e^{i\psi(y)} \int e^{i\theta(w)} dw$$

$$I = \left[e^{i\psi(y)} \int e^{i\theta(w)} dw\right] \left[e^{i\psi(y)} \int e^{i\theta(w)} dw\right]^*$$

$$I = \left\{\left[e^{i\psi(y)}\right]\left[e^{i\psi(y)}\right]^*\right\}\left\{\left[\int e^{i\theta(w)} dw\right]\left[\int e^{i\theta(w)} dw\right]^*\right\}$$

$$I = \left[\int e^{i\theta(w)} dw\right]\left[\int e^{i\theta(w)} dw\right]^*$$

4 Example 1: No atmosphere, no screen

This trivial example is to show the order in which we do things. First, we write the exponent out, with $\Phi(w) = 0$.

$$\theta(w) = \frac{\pi}{2}(w-y)^2$$

Then, we make the variable substitution to put things into a form for the Fresnel integrals:

$$z = w - y$$

and make the substitutions into the integral

$$dz = dw$$

Limits are $w = -\infty$ to ∞, so $z = -\infty - y$ to $\infty - y$, or $-\infty$ to ∞. This gives us

$$K(y) = \int_{-\infty}^{\infty} e^{i(\pi/2)z^2} dz = [C(\infty) + C(\infty)] + i\,[S(\infty) + S(\infty)] = 1 + i = \sqrt{i/2}$$

The intensity is $I = KK^*/2 = (1+i)(1-i)/2 = 1$, as it should.

5 Example 2: No atmosphere, half-screen

The seond example is almost as easy as the first, but reproduces the Fresnel fringe from a half screen.

First, we write the exponent out (as before), with $\Phi(w) = 0$

$$\theta(w) = \frac{\pi}{2}(w-y)^2$$

Then (as before), we make the variable substitution to put things into a form for the Fresnel integrals:

$$z = w - y$$

and make the substitutions into the integral

$$dz = dw$$

Limits are $w = y_s$ to ∞, so $z = y_s - y$ to ∞. This gives us

$$K(y) = \int_{y_s - y}^{\infty} e^{i(\pi/2)z^2} dz = [C(\infty) + C(y - y_s)] + i\,[S(\infty) + S(y - y_s)] = \left[\frac{1}{2} + C(y - y_s)\right] + i\left[\frac{1}{2} + S(y - y_s\right.$$

The intensity is $I = KK^*/2$is,

$$I = \frac{1}{2}\left\{\left[\frac{1}{2} + C(y - y_s)\right]^2 + \left[\frac{1}{2} + S(y - y_s)\right]^2\right\}$$

Since $y - y_s$ is the distance from the shadow edge, we see that at $y = -\infty$, I=0, and at $y = \infty$, I=1. As it should.

6 Example 3: Isothermal atmosphere, no screen

The next example shows what happens when we add an atmosphere, where the phase delay through the atmosphere is treated as a thin screen.

First, we write the exponent out. We Taylor expand Φ around w_1: $\Phi(w) = \phi_1 - (\phi_1/h)(w - w_1) + (\phi_1/2h^2)(w - w_1)^2$. What do we pick for w_1? Since (in geometric optics) the ray that passes through the phase screen at $w_1$is bent into the observer screen by $-\phi_1/\pi h = -(\phi_s/\pi h)\exp(-(w_1 - w_s)/h)$, the part of the phase screen that contributes the most to the integral at y is that part near w_1 , where $w_1$is defined as the solution to $y = w_1 - \phi_1/\pi h$ (recalling that $\phi$is a function of w). Or, if $\delta w = w - w_1$, then $w - y = \delta w + \phi_1/\pi h$, and

$$\theta(w) = \frac{\pi}{2}(\delta w + \phi_1/\pi h)^2 + \phi_1 - \frac{\phi_1}{h}\delta w + \frac{\phi_1}{2h^2}\delta w^2$$

Then, we make the variable substitution to put things into a form for the Fresnel integrals. We expand $\theta(w)$ and collect terms in δw

$$\theta(w) = \frac{\pi}{2}\left[\delta w^2 + \frac{2\phi_1}{\pi h}\delta w + \left[\frac{\phi_1}{\pi h}\right]^2 + \frac{\phi_1}{\pi h^2}\delta w^2 - \frac{2\phi_1}{\pi h}\delta w + \frac{2\phi_1}{\pi}\right]$$

$$\theta(w) = \frac{\pi}{2}\left[\left(1 + \frac{\phi_1}{\pi h^2}\right)\delta w^2 + \left[\frac{\phi_1}{\pi h}\right]^2 + \frac{2\phi_1}{\pi}\right]$$

Define $b_1 = \phi_1/\pi h^2$ (analogous to b, the bending parameter at the surface). We can also drop the bits that don't vary with w. This simplifies θ:

$$\theta(w) = \frac{\pi}{2}(1 + b_1)\delta w^2$$

After this slog, we can now make our variable substitution.

$$z = (1 + b_1)^{1/2}[w - w_1]$$

and make the substitutions into the integral

$$(1 + b_1)^{-1/2}dz = dw$$

Limits are $w = -\infty$ to ∞, so $z = -\infty$to ∞. This gives us

$$K(y) = \frac{\int_{-\infty}^{\infty} e^{i(\pi/2)z^2}dz}{\sqrt{1 + b_1}} = \frac{[C(\infty) + C(\infty)] + i\,[S(\infty) + S(\infty)]}{\sqrt{1 + b_1}} = \frac{1 + i}{\sqrt{1 + b_1}} = \frac{\sqrt{i/2}}{\sqrt{1 + b_1}}$$

The intensity is $I = KK^*/2 = 1/(1 + b_1)$, as it should.

Why "as it should"? It's easier to think of this from the $w$plane (the plane of the thin screen, not the plane of the observer). $y_1 = w_1 - \phi_1/\pi h$, and we've already related $-\phi_1/\pi h$ to $d\alpha = D\alpha/l$, the distance time the bending angle in units of l. This means that the ray passing though $w_1$in the screen also passes through y_1. Given a y_1, we're stuck with our standard Baum and Code problem of finding $w_1$iteratively, since we want to solve the following non-linear equation for w_1

$$y = w_1 - \frac{\phi_s e^{-(w-w_s)/h}}{\pi h}$$

where $\phi_s$is the phase delay at the surface, at screen position w_s.

But, given w_1, it's now easy to calculate $b_1 = \phi_1/\pi h^2$. Again, this is relatable to geometric optics, since $b_1 = -d\alpha/h = Dd\alpha/dr$, where $r = lw$ is the distance at the screen in "real" units (km etc).

7 Example 4: Isothermal atmosphere, half screen, above the surface

The next example is a trivial extension of the previous example. Again, we Taylor expand Φ around w_1, defining w_1 as the solution to $y = w_1 - \phi_1/\pi h$, $\delta w = w - w_1$, $b_1 = \phi_1/\pi h^2$ and drop the constant term in Φ.

$$\theta(w) = \frac{\pi}{2}(1 + b_1)\delta w^2$$

Make our variable substitution.

$$z = (1 + b_1)^{1/2}[w - w_1]$$

and make the substitutions into the integral

$$(1 + b_1)^{-1/2} dz = dw$$

Limits are $w = w_s$ to ∞, so $z = z_s$ to ∞, where

$$z_s = (w_s - w_1)\sqrt{1 + b_1}$$

This gives us

$$K(y) = \frac{\int_{z_s}^{\infty} e^{i(\pi/2)z^2} dz}{\sqrt{1 + b_1}} = \frac{[C(\infty) + C(-z_s)] + i\,[S(\infty) + S(-z_s)]}{\sqrt{1 + b_1}}$$

The intensity is

$$I(y) = \frac{1}{2(1 + b_1)} \left\{ \left[\frac{1}{2} + C(-z_s)\right]^2 + \left[\frac{1}{2} + S(-z_s)\right]^2 \right\}$$

For the airless case, this reduces to the usual equation for Fresnel diffraction by a half-screen.

For the case where the part of the atmosphere probed is far above the surface, $C(-z_s) = S(-z_s) = 1/2$, and this reduces to the geometric optics solution.

For the intermediate case, the airless fringe pattern is both fainter and spread out. It is fainter by the factor $1 + b_1$. It is spread out because a change of dy at the observer plane is a change of $(1 + b_1)dy$ in w_1.

8 Example 5: Isothermal atmosphere, half screen, below the surface

If $y < y_s$then we can't solve $y = w_1 - \phi_1/\pi h$ for $w_1$above the surface. The part of the atmosphere that most affects the Fresnel integral is the part just above the atmosphere. So, Taylor expand Φ around w_s, the location of the surface in the screen plane. The (geometric-optics) surface in the observer plane is $y_s = w_s - \phi_s/\pi h$. In FG76, their ϕ_0 is the same as this ϕ_s, their $y = 0$ is our $y = y_s$, and their $w = y_0$ is our $w = w_s$. Defining $\delta w = w - w_s$, we get

$$w - y = \delta w - (y - w_s)$$

$$\phi(w) \approx \phi_s - \frac{\phi_s}{h}\delta w + \frac{\phi_s}{2h^2}\delta w^2$$

$$\theta(w) = \frac{\pi}{2}(\delta w - (y - w_s))^2 + \phi_s - \frac{\phi_s}{h}\delta w + \frac{\phi_s}{2h^2}\delta w^2$$

Expand in δw:

$$\theta(w) = \frac{\pi}{2}\left[\delta w^2 - 2(y - w_s)\delta w + (y - w_s)^2 + \frac{\phi_s}{\pi h^2}\delta w^2 - \frac{2\phi_s}{\pi h}\delta w + \frac{2\phi_s}{\pi}\right]$$

Collect terms and substitute $y_s$for $w_s - \phi_s/\pi h$ in the linear term.

$$\theta(w) = \frac{\pi}{2}\left[(1 + b_s)\delta w^2 - 2(y - y_s)\delta w + (y - w_s)^2 + \frac{2\phi_s}{\pi}\right]$$

"Complete the square" and drop the constant term.

$$\theta(w) = \frac{\pi}{2}(1 + b_s)\left[\delta w - \frac{y - y_s}{1 + b_s}\right]^2$$

Make our variable substitution.

$$z = (1 + b_s)^{1/2}\left[w - w_s - \frac{y - y_s}{1 + b_s}\right]$$

and make the substitutions into the integral

$$(1 + b_1)^{-1/2}dz = dw$$

Limits are $w = w_s$to ∞, so $z = z_s$to ∞, where

$$z_s = (y_s - y)/\sqrt{1 + b_s}$$

This gives us

$$K(y) = \frac{\int_{z_s}^{\infty} e^{i(\pi/2)z^2} dz}{\sqrt{1+b_s}} = \frac{[C(\infty) + C(-z_s)] + i\,[S(\infty) + S(-z_s)]}{\sqrt{1+b_s}}$$

The intensity is

$$I(y) = \frac{1}{2(1+b_s)} \left\{ \left[\frac{1}{2} + C(-z_s)\right]^2 + \left[\frac{1}{2} + S(-z_s)\right]^2 \right\}$$

For the airless case, this reduces to the usual equation for Fresnel diffraction by a hald-screen.

For the case where the observer is below the geometric surface, $C(-z_s) = S(-z_s) = -1/2$, and this reduces to zero flux.

9 Example 6: Isothermal atmosphere, half screen, above the surface (again)

In example 4, we expanded the phase delay around w_1. But, the diffraction arises because of the blockage of the field for $w < w_s$. So, maybe we should be doing the following:

$$K(y) = K_{geom}(y) + K_{diff}(y)$$

$$K_{geom}(y) = \int_{-\infty}^{\infty} e^{i\theta(w)} dw$$

$$K_{diff}(y) = -\int_{-\infty}^{w_s} e^{i\theta(w)} dw = \int_{w_s}^{-\infty} e^{i\theta(w)} dw$$

To find K_{geom}, follow example 3 (expanding around w_1) to get

$$K_{geom}(y) = \frac{1+i}{\sqrt{1+b_1}}$$

To find K_{diff}, follow example 5 to get

$$K_{diff}(y) = \frac{\int_{z_s}^{-\infty} e^{i(\pi/2)z^2} dz}{\sqrt{1+b_s}} = \frac{\left[C(-z_s) - \frac{1}{2}\right] + i\left[S(-z_s) - \frac{1}{2}\right]}{\sqrt{1+b_s}}$$

Putting it all together, we get

$$I(y) = \frac{1}{2(1+b_s)} \left\{ \left[\sqrt{\frac{1+b_s}{1+b_1}} - \frac{1}{2} + C(-z_s)\right]^2 + \left[\sqrt{\frac{1+b_s}{1+b_1}} - \frac{1}{2} + S(-z_s)\right]^2 \right\}$$

Again, for no atmosphere ($b_s = 0$, $b_1 = 0$, $-z_s = y - y_s$), this reduces to the airless case. For a deep atmosphere ($-z_s$ large), this reduces to the geometric solution,

10 Comparison with French and Gierasch 1976

This is close to, but not quite, FG76, who have

$$I \approx \frac{1}{2(1+b)} \left\{ \left[C(z) - \frac{1}{2} \right]^2 + \left[S(z) - \frac{1}{2} \right]^2 \right\}$$

$$z = \frac{\pi y^2}{2(1+b)}$$

First, the intensity should be addint 1/2, not subtracting. Second, the equation for $z = \ldots$ was probably meant to be $z^2 = \ldots$. Finally, the factor of $\pi/2$ is superfluous.

11 Actually using the isothermal atmosphere, half-screen

Taking this back into the realm of real units:

We could work straight in distances, not times. We have a lightcurve with shadow-plane radii ρ_i(km). We know the distance, D (km) and the wavelength λ(oddly, in km), giving us the Fresnel scale l (km). Our model takes the scale height H (km), the surface radius

We have a lightcurve with times t_i. We think we know the wavelength and the perpendicular sky velocity, giving us $t_F = \sqrt{\lambda D/2}/v_\perp$, the Fresnel time scale. The